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PART II. PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

1. The Cauchy-Kowalewski Theorem. As in Part I, our work depends directly on the Cauchy-Kowalewski theorem for equations of the second order.* Let us consider a partial differential equation of the second order in one dependent and two independent variables,

$$F[x, y, z, p, q, r, s, t] = 0 \tag{1}$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$; and let this equation be solved for r,

$$r = f[x, y, z, p, q, s, t]. \tag{2}$$

THEOREM: *Then there exists one and only one solution*

$$z = v(x, y)$$

which satisfies the boundary condition,

$$z\Big]_{x=x_0} = \phi(y), \qquad p\Big]_{x=x_0} = \psi(y) \tag{3}$$

where $\phi(y)$ *and* $\psi(y)$ *are each preassigned analytic functions of* y *near* $y = y_0$, *provided that* $f[x, y, z, p, q, s, t]$ *is a single valued analytic function of* x, y, z, p, q, s, t *near the point* $x = x_0$, $y = y_0$, $z = z_0 = \phi(y_0)$, $p = p_0 = \psi(y_0)$, $q = q_0 = \phi'(y_0)$, $s = s_0 = \psi'(y_0)$, $t = t_0 = \phi''(y_0)$, *and* $v(x, y)$ *is analytic near* (x_0, y_0); *where* $\phi'(y) = d\phi(y)/dy$, $\phi''(y) = d^2\phi(y)/dy^2$, *etc.* The equation (2) is said to be in *normal form*, if this condition is satisfied. The equation (1) can be put in the form (2) where $f(x, y, z, p, q, s, t)$ is analytic at the point specified, provided $F(x, y, z, p, q, r, s, t)$ is analytic, and $\partial F/\partial r \neq 0$, at the point $(x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0)$, where $F[x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0] = 0$.

Geometrically the theorem states that one and only one analytic integral surface of (1) (corresponding to each single valued solution of the form (2)) exists, upon which lies a given *strip* (curve with appended tangent planes) of which the curve lies in a plane parallel to the plane of YOZ, provided only that F is analytic along the given strip in the manner specified above.

Following Part I, we will then try to find those *strips* in space for which the equation (1) cannot be put into normal form, and for which, for this reason, the Cauchy-Kowalewski theorem may fail.

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* See footnotes, p. 121.

2. Characteristic Strips on a given Integral Surface. Let there be given an integral surface

$$z = \nu(x, y) \tag{4}$$

where $\nu\ (x, y)$ is a known analytic function of x and y. We will first determine those strips which lie on this surface, and for which the Cauchy-Kowalewski theorem may fail. Let $y = \lambda(x)$ be the equation of a curve on the surface (4) where $\lambda(x)$ is an analytic function of x. The strip determined by this curve, and the tangent planes to the surface (4) at each point of the curve is then given by the equations

$$\left\{\begin{aligned} y &= \lambda(x) \\ z &= \nu(x, y) \\ p &= \frac{\partial \nu(x, y)}{\partial x} \\ q &= \frac{\partial \nu(x, y)}{\partial y} \end{aligned}\right. \qquad \text{or} \qquad \left\{\begin{aligned} y &= \lambda(x) \\ z &= \nu(x, \lambda(x)) &&= \mu(x) \\ p &= \left.\frac{\partial \nu(x, y)}{\partial x}\right]_{y=\lambda(x)} &&= \phi_1(x) \\ q &= \left.\frac{\partial \nu(x, y)}{\partial y}\right]_{y=\lambda(x)} &&= \phi_2(x). \end{aligned}\right. \tag{5}$$

Let us now transform the equation (1) by the transformation

$$\begin{cases} x_1 = y - \lambda(x) \\ y_1 = x \end{cases} \qquad \text{or} \qquad \begin{cases} x = y_1 \\ y = x_1 + \lambda(y_1) \end{cases} \tag{6}$$

by means of which p and q are transformed as follows

$$\begin{cases} p_1 = q \\ q_1 = q\lambda'(y_1) + p \end{cases} \qquad \text{or} \qquad \begin{cases} p = p_1\lambda(x) + q_1 \\ q = p_1 \end{cases} \tag{7}$$

where $\lambda'(x) = d\lambda(x)/dx$; and the strip (5) goes into the strip

$$\begin{cases} x_1 = 0 \\ z = \mu(y_1) \end{cases} \qquad \begin{aligned} p_1 &= \phi_2(y_1) \\ q_1 &= \phi_2(y_1)\lambda'(y_1) + \phi_1(y_1) = \mu'(y_1) \end{aligned} \tag{8}$$

which is of the form (3) to which the Cauchy-Kowalewski theorem applies, where $\mu(y_1)$ and $\phi_2(y_1)$ are any analytic functions of y_1, and $\mu'(y_1) = d\mu(y_1)/dy_1$. The second derivatives r_1, s_1, t_1 are given by the equations:

$$(9)\quad \left\{\begin{aligned} r_1 &= \frac{\partial^2 z}{\partial x} &&= t \\ s_1 &= \frac{\partial^2 z}{\partial x_1\,\partial y_1} &&= s + t\lambda'(y_1) \\ t_1 &= \frac{\partial^2 z}{\partial y_1^2} &&= r + 2s\lambda' + t\lambda'^2 + q\lambda'' \end{aligned}\right. \qquad \text{or} \qquad \left\{\begin{aligned} r &= r_1\lambda'^2 - 2s_1\lambda' + t_1 - p_1\lambda'' \\ s &= s_1 - r_1\lambda' \\ t &= r_1 \end{aligned}\right.$$

where $\lambda' = d\lambda(x)/dx = d\lambda(y_1)/dy_1$, etc.; and hence the transformed differential equation is

$$(10)\quad F[y_1\,;x_1 + \lambda(y_1)\,;z\,;q_1 + p_1\lambda\,;p_1\,;r_1\lambda'^2 - 2s_1\lambda' + t_1 - p_1\lambda''\,;s_1 - r_1\lambda'\,;r_1] = 0.$$

Hence if F is analytic and $\partial F/\partial r_1 \neq 0$ near the point $x_1 = 0$, $y_1 = y_1$, $z = \mu(y_1)$, $p_1 = \phi_2(y_1)$, $q_1 = \mu'(y_1)$, $r_1 = \chi(y_1)$, $s_1 = \phi_2'(y_1)$, $t_1 = \mu''(y_1)$, where $\chi(y_1)$ is such that (10) is satisfied, then the strip (5), by the theorem, determines uniquely an analytic integral surface. Or, as in §2, Part I, we may transform back into the old variables and substitute $\nu(x, y)$ throughout for z, and then $\lambda(x)$ for y, which will be equivalent to the above substitution since (4) is a solution of (1), and since (7) and (9) give the same values for p, q, r, s, t as are obtained from $p = \nu_x[x, \lambda(x)]$, $q = \nu_y[x, \lambda(x)]$, $r = \nu_{xx}[x, \lambda(x)]$, etc.

But if

$$(11)\qquad \frac{\partial F}{\partial r_1} \equiv \frac{\partial F}{\partial r}\left[\frac{d\lambda(x)}{dx}\right]^2 - \frac{\partial F}{\partial s}\,\frac{d\lambda(x)}{dx} + \frac{\partial F}{\partial t} = 0$$

when $\nu(x, y)$ is substituted for z, and then $\lambda(x)$ is substituted for y, the Cauchy-Kowalewski theorem may fail for the strip (5). If then $\nu(x, y)$ is substituted throughout for z in (11), this equation becomes an ordinary differential equation of the first order and second degree for y as a function of x,

$$(11a)\qquad R(x, y)\left(\frac{dy}{dx}\right)^2 - S(x, y)\,\frac{dy}{dx} + T(x, y) = 0,$$

where R, S, T are the functions of x and y obtained from $\partial F/\partial r$, $\partial F/\partial s$, $\partial F/\partial t$ respectively, when $\nu(x, y)$ is substituted for z. Any solution of this equation $y = \lambda(x)$ defines a strip (5) for which the Cauchy-Kowalewski theorem may fail. Any curve

$$(12) \qquad \begin{cases} y = \lambda(x) \\ z = \nu(x, y) \end{cases}$$

which lies on an integral surface (4) will be called a *characteristic curve*, if $y = \lambda(x)$ is a solution of (11a).* The corresponding strip(5) will be called a *characteristic strip*, where the tangent plane attached to the curve at each point is the tangent plane of the surface (4) at that point.

It is clear that there will be two one parameter families of characteristic curves, in general, on every integral surface, of which two curves and but two will pass through any given point of the surface. These two one parameter families correspond to the two solutions of (11a), as an algebraic equation for dy/dx. The two families will be both *real*, both *imaginary*, or *coincident*, according as $S^2 - 4RT > 0, < 0, = 0$, on the surface (4). The equation (1) is called a *hyperbolic*, *elliptic*, or *parabolic* equation on the surface (4), or in any region upon it, in these three cases respectively.

Any other curve of the type (12) on the surface (4) together with its strip determined by (5) evidently determines the integral surface (4) uniquely, provided F is analytic along the strip.

We have tacitly assumed that $R \not\equiv 0$ identically in x and y on the surface (4). If $R = 0$, we can evidently do the same work by interchanging x and y throughout unless T also vanishes identically on the surface (4), or by replacing x and y by $x - y$ and $x + y$ respectively, unless $S \equiv 0$ on the surface(4). If $R \equiv 0$, $T \equiv 0$, and $S \equiv 0$ (so that $RT - S^2 \equiv 0$) on the surface (4), we will call (4) a *singular integral surface*,† and we will exclude such a surface explicitly in the statement of all our theorems, for it is evident that any curve on such a surface would satisfy the equation (11a), and that for it the Cauchy-Kowalewski theorem might fail.

It is evident that the same equation (11a) after dy/dx is replaced by $[dx/dy]^{-1}$ determines whether a curve of the form

$$(13) \qquad \begin{cases} x = x_0 \\ z = \nu(x, y) \end{cases} \quad \text{or} \quad \begin{cases} x = x_0 \\ z = \nu(x_0, y) = \mu(y) \end{cases}$$

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* Equation (11a) will be recognized as the principal equation of the characteristics, as ordinarily derived. See Sommerfeld, *Ency. d. Math. Wiss.*, II, A, 7, p. 598; von Weber, *l. c.*, p. 366; Goursat, *l. c.*, *2 ordre*, I, p. 172.

† Compare p. 127, Art. 2; and see Darboux, *l. c.*

together with its corresponding strip, satisfies the requirements of the Cauchy-Kowalewski theorem, if F is analytic along the strip. For the requirement is precisely that $R \neq 0$ along the strip, and the equation (11a) reduces to this form for a curve of the above type.

3. The Characteristic Strips in Space. If the equation of the integral surface (4) is unknown, we shall have to determine $\mu(x)$, $\phi_1(x)$, $\phi_2(x)$ as well as $\lambda(x)$, in order to determine the characteristic strip (5). The values of y, z, p, q, r, s, t, as functions of x which satisfy (11) will then be given by $y = \lambda(x)$, $z = \mu(x)$, $p = \phi_1(x)$, $q = \phi_2(x)$, $r = \phi_3(x)$, $s = \phi_4(x)$, $t = \phi_5(x)$, say, for which the following several relations (13) must hold. First,

$$F[x,\ \lambda(x),\ \mu(x),\ \phi_1(x),\ \phi_2(x),\ \phi_3(x),\ \phi_4(x),\ \phi_5(x)] = 0 \tag{13a}$$

since the quantities substituted in (11) are to satisfy (1). Moreover since $\nu(x,y)$ is a solution of (1) we have

$$F[x;\ y;\ \nu(x,y);\ \nu_x;\ \nu_y;\ \nu_{xx};\ \nu_{xy};\ \nu_{yy}] = 0 \tag{13b}$$

identically in x and y. Hence the total derivatives of F with respect to x and y formed from this equation vanish identically in x and y. If we introduce the notation

$$\left\{\begin{aligned} X &= \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + r\frac{\partial F}{\partial p} + s\frac{\partial F}{\partial q} \\ Y &= \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} + s\frac{\partial F}{\partial p} + t\frac{\partial F}{\partial q} \end{aligned}\right. \tag{13c}$$

we may write these relations as follows:

$$\left\{\begin{aligned} \frac{dF}{dx} &= X + \alpha R + \beta S + \gamma T = 0 \\ \frac{dF}{dy} &= Y + \beta R + \gamma S + \delta T = 0 \end{aligned}\right. \tag{13d}$$

where $\alpha = \partial^3 z/\partial x^3$, $\beta = \partial^3 z/\partial x^2 \partial y$, $\gamma = \partial^3 z/\partial x \partial y^2$, $\delta = \partial^3 z/\partial y^3$. We have also the identities

$$\left\{\begin{aligned} z' &= p + qy' \qquad & r' &= \alpha + \beta y' \\ p' &= r + sy' & s' &= \beta + \gamma y' \\ q' &= s + ty' & t' &= \gamma + \delta y' \end{aligned}\right. \tag{13e}$$

where $y' = dy/dx$, $z' = dz/dx$, etc., which are identically satisfied in x, when $\nu(x,y)$ is substituted for z, and then $\lambda(x)$ is substituted for y. Eliminating

α, β, γ, δ between these ten equations (13) and combining with (11a) which λ, μ, ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5 must also satisfy, we obtain for these functions the following equations

$$(14)\quad\begin{cases}(a)\quad F[x,\lambda(x),\mu(x),\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x),\phi_5(x)]=0\\ (b)\quad Ry'^2-Sy'+T=0\\ (c)\quad z'=p+qy'\\ (d)\quad p'=r+sy'\\ (e)\quad q'=s+ty'\\ (f)\quad X+r'R+S'(S-Ry')=0\\ (g)\quad Y+s'R+t'\ (S-Ry')=0.\end{cases}$$

These seven equations are however not independent, for we obtain the same result if we differentiate (14a) with respect to x totally, or if we multiply (14g) by y' and add it to (14f). The equations (14) then constitute a set of six independent ordinary differential equations for the seven variables y, z, p, q, r, s, t as functions of x. If one of them, say $y=\lambda(x)$, be chosen at pleasure, the remainder will be given uniquely if we further assign their values for any constant value of x. We will call a set of solutions of (14) a *characteristic torsion strip*,* since the torsion (r, s, t) as well as the tangent plane (p, q) will be determined at each point (x, y, z) of the curve. Hence follows the

THEOREM: *Given any preassigned set of constant values of the solutions,* x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0, *that is, a fixed torsion element at a point* x_0, y_0, z_0, *we have one and only one characteristic torsion strip through this element, corresponding to any choice of a certain arbitrary function. The simple strip contained in this characteristic torsion strip will be a characteristic strip, and the curve contained in this will, in turn, be a characteristic curve.* But more than one characteristic strip may be arranged along a given characteristic curve, and more than one characteristic torsion strip along each characteristic strip, although, *in general*, the characteristic curves, strips and torsion strips will be paired in a one-to-one manner.

For the linear equation

$$(15)\qquad A(x,y)r+B(x,y)s+C(x,y)t=D(x,y,z,p,q)$$

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* See, *e. g.* Goursat, *l. c. 2 ordre*, I, p. 173, etc.

the equation (11) becomes

$$(11b)\qquad A(x, y)\left(\frac{dy}{dx}\right)^2 - B(x, y)\frac{dy}{dx} + C(x, y) = 0$$

which does not contain z, p, q, r, s, t. This case is then especially simple, though simplifications also arise in more general cases. Here it is not necessary to substitute for z, p, q, r, s, t, as in general. There are two one parameter families of solutions of (11b) of the form $y = \lambda(x)$. Any strip in space whose projection on the (x, y) plane is one of these curves, and which lies on an integral surface, is a characteristic strip. It is easy to show that the equations (14b), (14c), (14d), (14e) can be solved separately for z, p, q in this case. For having found $y = \lambda(x)$, a solution of (11b), it results from an algebraic calculation that we may replace the first five of equations (14) by the equations (15), (11b), and the equations

$$\begin{cases} z' = p + qa \\ p' = -\dfrac{C}{a}q' + \dfrac{D}{2A} \end{cases} \qquad \left(a = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}\right)$$

and these equations contain neither r nor s. If $z = \mu(x)$ be taken at pleasure, these two equations can be solved to find $p = \phi_1(x)$, $q = \phi_2(x)$, involving, in general, one arbitrary constant. *Hence along any curve whose projection is one of the two parameter families of solutions of* (11b) *there exist, in general, a one parameter family of characteristic strips of the equation* (15).

From this point on, the development of the theory would follow exactly the ordinary course. We should next prove the two theorems:

(1) *Every integral surface of* (1) *containing a given torsion element* $(x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0,)$ *and a certain characteristic curve on which the given torsion element lies, also contains the whole characteristic torsion strip determined by the given torsion element and the given characteristic curve, provided that* $S^2 - 4RT \not\equiv 0$ *along the given characteristic curve.* Or, any two integral surfaces through the same characteristic curve, having double contact at one point of that curve, have double contact along its entire length, provided that $S^2 - 4RT \not\equiv 0$ along the curve, which special case is left for later investigation.

(2) *A characteristic strip actually does not determine uniquely the integral surface on which it lies: — on the contrary, there are an infinite number of integral surfaces through this same strip; and in fact* (provided $S^2 - 4RT \not\equiv 0$ along the curves used) *if two characteristic curves belonging to different families pass through the same torsion element, one and only one integral surface exists which passes through the two curves.*

The proof of these two theorems, as well as the remainder of the ordinary theory, we will pass over in silence. The statement of these theorems was, however, essential to establish our point of view of the characteristics, *as those curves which, together with their appended tangent planes, actually do not determine uniquely the integral surface on which they lie.*

It results from a further development that the theory of characteristics is valuable in the actual solution of problems, and whole classes of equations can be treated by methods based on it, so as eventually to effect their complete solutions.†

But it is clear that we cannot proceed to reduce the solution of the partial differential equation of the second order (1) to the solution of a set of ordinary differential equations, as we did in Part I for a partial differential equation of the first order. For the equations (14) comprise only six independent equations for the seven variables y, z, p, q, r, s, t as functions of x, and these are all the relations which can be obtained.

4. The Integral Strips. Let us now seek, as in Part I, §4, all those strips in space for which the Cauchy-Kowalewski theorem may fail in the above sense. Such strips we will call integral strips. It is evident that the characteristic strips will be included among the integral strips, and that an integral strip which is not a characteristic strip does not lie on any integral surface.

Since we do not assume that the strip

$$(16) \qquad \begin{cases} y = \lambda(x) \\ z = \mu(x) \end{cases} \qquad \begin{array}{l} p = \phi_1(x) = \mu'(x) - \phi_2(x)\lambda'(x) \\ q = \phi_2(x) \end{array}$$

lies on an integral surface, it will be necessary to return to the substitutions for y, z, p, q, r, s, t, in Art. 2, in order to determine the substitutions which we must make in (11) for y, z, p, q, r, s, t. These are given by the substitutions (6), (7), (9) in the following form:

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* See Goursat, *l. c.*, *2 ordre*, II, pp. 184-5.

† See Goursat, *l. c.*, *2 ordre*, I and II.

$$(17)\quad \begin{cases} x = y_1 & = x \\ y = x_1 + \lambda(y_1) & = 0 + \lambda(x) \\ z = z & = \mu(x) \\ p = q_1 - \lambda' p_1 & = \mu'(x) - \phi_2(x)\,\lambda'(x) \\ q = p_1 & = \phi_2(x) \\ r = r_1\lambda'^2 - 2s_1\lambda' + t_1 - p_1\lambda'' & = \chi(x)\lambda'^2 - 2\phi'_2(x)\lambda'(x) + \mu''(x) - \phi_2(x)\lambda''(x) \\ s = s_1 - r_1\lambda' & = \phi'_2(x) - \chi(x)\,\lambda'(x) \\ t = r_1 & = \chi(x) \\ F(x, y, z, p, q, r, s, t) & = 0 \end{cases}$$

since we were to substitute $x_1 = 0$, $y_1 = y_1$, $z = \mu(y_1)$, $p_1 = \phi_2(y_1)$, $q_1 = \mu'(y_1)$, $r_1 = \chi(y_1)$, $s_1 = \phi_2'(y_1)$, $t_1 = \mu''(y_1)$; where $\chi(y_1)$ is determined by the condition that (1) be satisfied in the transformed variables.

The conditions (17) will be satisfied by the strip (16) if

$$(18)\quad \begin{cases} (a) & F(x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x), r, s, t,) = 0. \\ (b) & \dfrac{d\mu(x)}{dx} = \phi_1(x) + \phi_2(x)\,\dfrac{d\lambda(x)}{dx} \\ (c) & \dfrac{d\phi_1(x)}{dx} = r + s\,\dfrac{d\lambda(x)}{dx} \\ (d) & \dfrac{d\phi_2(x)}{dx} = s + t\,\dfrac{d\lambda(x)}{dx} \end{cases}$$

which are the only conditions upon the functions of x which are to be substituted in (11) for y, z, p, q, r, s, t. In general r, s, and t may be eliminated between (11) and (18) so as to obtain an equation

$$(19)\qquad M\left(x, y, z, p, q, \frac{dy}{dx}, \frac{dp}{dx}, \frac{dq}{dx}\right) = 0.$$

This equation and (18b) will be the only equations which the strip (5) must satisfy in order to be an integral strip and we will therefore, by analogy to Part I, call (19) the *Monge equation* of the equation (1).

In general any two of the functions $\lambda(x)$, $\mu(x)$, $\phi_1(x)$, $\phi_2(x)$, may be chosen at pleasure, and the other two found from (18b) and (19). Thus if $\lambda(x)$ and $\mu(x)$ are given, (19) reduces, in general, by means of (18b), to a single ordinary differential equation for $\phi_1(x)$, say. Then $\phi_2(x)$ is determined from $\phi_1(x)$ by (18b). *Hence there will, in general, be a one parameter family of integral strips through any curve in space*; but this may not be true in a particular example on account of the special form which the equations (18) may assume.

For instance, for the linear equation (15) the equation (11) becomes (11b), and it is not necessary to substitute for z, p, q, r, s, t, since only x and y occur in the equation. Having obtained, as above, the two one parameter families of curves in the XOY plane as solutions of (11b), it is clear that any strip of the form (16) will be an integral strip, if and only if $y = \lambda(x)$ is among these curves. *Hence any strip whatever through any curve in space whose projection is a solution of* (11b) *will be an integral strip of the equation* (15); *but no integral strip passes through any other curve in space.*

Returning to the general case of the equation (1), we obtain directly from the theorems of §3 the following

THEOREM: *No integral strip determines uniquely an integral surface, there being more than one integral surface through every characteristic strip, and none at all through any other integral strip.* For, by derivation, the characteristic strips are all those integral strips which lie on integral surfaces; and hence no other integral strip can possibly lie on any integral surface.

Let us then consider any strip S of the form (16), which is not an integral strip. It is clear from the Cauchy-Kowalewski theorem, and from §2, that such a non-integral strip will determine uniquely an analytic integral surface through itself, for any portion of S for which $F(x, y, z, p, q, r, s, t)$ is analytic, and for which (11) does not hold at any point $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$ of the interval, where r_0, s_0, t_0 are determined from (16) by (17) or (18). As in Part I, §4, we see that (11) cannot hold except at isolated points on S, since S is supposed not to be an integral strip. At the exceptional points where (11) is satisfied, S (with its strip) is necessarily tangent to some integral strip. For, by hypothesis, the given values of x, y, z, p, q determine at that point values of r_0, s_0, t_0 which, together with x_0, y_0, z_0, p_0, q_0, satisfy (11). Or, the values of $x_0, y_0, z_0, p_0, q_0, dy/dx]_0, dz/dx]_0, dp/dx]_0, dq/dx]_0$, obtained from S satisfy (19) and (18b); and it is clear that such a set of constant values, which satisfy (18b) and (19), determines in general one or more solutions of (18b) and (19), or of the equations (11) and (18) from which these are derived; *i. e.*, one or more integral strips.* Any one of these integral strips is then tangent to S at the point x_0, y_0, z_0, in the sense that each of the torsion elements through the surface element x_0, y_0, z_0, p_0, q_0, in which the strip S lies also contains the integral strip in question.† For if the values of $\lambda'(x)$, $\mu'(x)$, $\phi'_2(x)$ coincide at x_0, y_0, z_0, p_0, q_0, then it appears from (17) that the values of r and s will coincide at that point, for *any arbitrary choice* of t (not necessarily satisfying (1)).

Excluding these isolated points on S where S is tangent to some integral strip, *i. e.* where $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$, determined from (16) by (17) or (18) satisfy (11), any remaining portion of S uniquely determines an analytic integral surface through itself, provided F is analytic at every point of this portion of S. As in Part I, §4, *if S lies wholly on an analytic integral surface, it determines that whole surface uniquely, irrespective of the above peculiar points.* And the only other cases which can arise, aside from a mixture of the three, are those in which (1) *S uniquely determines a single integral surface, which is analytic except at the excluded points*; or where (2) *the portions of S lying between excluded points determine pieces of different analytic integral surfaces.* It is clear, for example, that if two analytic integral surfaces be given which have a common characteristic strip (§3), then a non-integral analytic strip S may be drawn of which one portion lies on one of the given integral surfaces, while another portion lies on the other surface; the strip S being tangent to the given characteristic strip at the point where S crosses from one surface to the other. Such a non-integral strip evidently does not uniquely determine a single analytic integral surface along its whole extent, though each separate portion does uniquely determine an integral surface through itself. But it does determine uniquely an integral surface which is continuous and has a continuously turning tangent plane at the excluded point, and which is elsewhere analytic in the neighborhood of the strip, provided F is analytic along S in the interval considered. We may then state the

THEOREM: *Any strip S in space, of the form* (16), *which is not an integral strip, determines uniquely an integral surface which is analytic and which contains the given strip, for any portion of S in which no point lies at which S*

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*Among these there will be, in general, one or more characteristic strips. Similarly in Part I, §4, there will be among the analogous integral curves in general *one and only one characteristic curve*; and the points to be excluded can be said with equal propriety to be those in which the curve in question (Part I) is tangent to a *characteristic curve*.

†The similar definition of tangency of two *curves* might have been used in Part I, §4: two curves are tangent at any point when each of the surface elements through that point, in which one of the curves lies, also contains the other curve.

is tangent to an integral strip, or at which $F(x, y, z, p, q, r, s, t,)$ *is non-analytic; the point* $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$ *being found from* (16) *by means of* (17) *or* (18). *But in any case any portion of* S *in which* F *is analytic at every point, uniquely determines an integral surface which is in general analytic, and which is at least continuous at the points where* S *is tangent to an integral curve; and the integral surface so determined may be analytic even at such points.*

It is clear that the above work, and in particular this last theorem, holds, with a suitable change of notation, for a strip of the form

$$\begin{cases} x = x_0 \\ z = \mu(y) \end{cases} \qquad \begin{matrix} p = \psi(y) \\ q = \phi'(y) \end{matrix}$$

which is of the type (3) for which the Cauchy-Kowalewski theorem was stated in §1.

For the linear equation (15) we have seen that the integral strips were all those strips in space which project into any curve of the two one parameter families of solutions of (11*b*). And we saw that, among these, the characteristic strips constitute in general a one parameter family along any space curve which projects into a solution of (11*b*). Any strip S in space, along a curve K, will be tangent (as a strip) to an integral strip, if and only if K is tangent to a curve which projects into a solution of (11*b*). In particular the strip S will certainly not be tangent to an integral strip if the projection of K on the XOY plane is not tangent to a solution of (11*b*). Hence we obtain for the linear equation (15) the following

THEOREM: *Given a curve* C *which projects into a solution of* (11*b*), *there is in general a one parameter family of characteristic strips through* C, *through each of which passes more than one integral surface of* (15). *Aside from these characteristic strips, no strip through* C *lies on any integral surface whatever, since all strips through* C *are integral strips. Any strip* S *in space along a curve* K, *which does not project into a solution of* (11*b*), *uniquely determines an analytic integral surface, for any portion of* S *in which no point lies at which* F *is non-analytic, or at which* K *is tangent to a curve* C *which projects into a solution of* (11*b*).

On account of the importance of the linear equation (15), and on account of the special rôle which the solutions of (11*b*) play in the theory, the two one parameter families of solutions of (11*b*) are themselves often called the "characteristics" of (15).

5. Relation to the Calculus of Variations.* As in the case of equations of the first order, for equations of the second order also, we find an intimate relation between the theory of characteristics and the Calculus of Variations.

Given the problem to render the integral

$$I = \int z' d\tau \tag{1}$$

a minimum, under the auxiliary conditions

$$F(x, y, z, p, q, r, s, t) = 0 \tag{2}$$

$$z' = px' + qy' \tag{3}$$

$$p' = rx' + sy' \tag{4}$$

$$q' = sx' + ty' \tag{5}$$

where $z' = \partial z / \partial t$, etc., τ being a parameter, while x, y, z, p, q, r, s, t, F have the same meaning as in the last article. We transform the given problem again into the problem, to render the following integral a minimum, under no auxiliary conditions:

$$I' = \int \Big[z' + \lambda(z' - px' - qy') + \mu(p' - rx' - sy') + \nu(q' - sx' - ty') + \pi F \Big] d\tau$$

where λ, μ, ν, π represent undetermined functions of τ. We have as necessary conditions for the solution of the problem

$$(6) \left\{ \begin{array}{lll} (a) & \dfrac{d}{d\tau}(-p\lambda - \mu r - \nu s) & = \pi \dfrac{\partial F}{\partial x} \\ (b) & \dfrac{d}{d\tau}(-q\lambda - \mu s - \nu t) & = \pi \dfrac{\partial F}{\partial y} \\ (c) & \dfrac{d}{d\tau}(\lambda) & = \pi \dfrac{\partial F}{\partial z} \\ (d) & \dfrac{d}{d\tau}(\mu) & = -\lambda x' + \pi \dfrac{\partial F}{\partial p} = -\lambda x' + \pi P \\ (e) & \dfrac{d}{d\tau}(\nu) & = -\lambda y' + \pi \dfrac{\partial F}{\partial q} = -\lambda y' + \pi Q \\ (f) & -\mu x' + \pi R & = 0 \\ (g) & -(\mu y' + \nu x') + \pi S & = 0 \\ (h) & -\nu y' + \pi T & = 0 \end{array} \right.$$

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* For the statements made in this article see Kneser, *Lehrbuch der Variationsrechnung*.

From (6c), $\frac{\lambda'}{\pi} = F_z$; from (6$f$), $\mu = \frac{\pi R}{x'}$; and from (6h), $\nu = \frac{\pi T}{y'}$. Hence (6g) becomes

$$\frac{\pi R}{x'} y' + \frac{\pi T}{x'} y' - \pi S = 0,$$

or

$$R\left(\frac{dy}{dx}\right)^2 - S\frac{dy}{dx} + T = 0, \tag{7}$$

which coincides with equation (11) of the preceding article, the fundamental characteristic equation. Moreover, (6a) becomes on expansion

$$p\lambda' + p'\lambda + \mu r' + \mu' r + \nu s' + \nu' s + \pi F_x = 0. \tag{8}$$

Substituting from (6f) and (6h) and also from (6b), (6d), and (6e) we have

$$p\pi F_z + p'\lambda + r'\frac{\pi R}{x'} + r(\pi P - \lambda x') + s'\frac{\pi T}{y'} + s(\pi Q - \lambda y') + \pi F_x = 0,$$

or

$$\pi\left[pF_z + \frac{r'R}{x'} + Pr + \frac{Ts'}{y'} + Qs + F_x\right] + \lambda\,[p' - rx' - sy'] = 0.$$

But $p' - rx - sy' = 0$ by (4); and since $X = F_x + F_z p + Qs + P\gamma$, we have

$$X + R\frac{dr}{dx} + \frac{\frac{ds}{dx}T}{\frac{dy}{dx}} = 0. \tag{9}$$

Likewise from (6b) we obtain

$$Y + R\frac{ds}{dx} + \frac{\frac{dt}{dx}T}{\frac{dy}{dx}} = 0. \tag{10}$$

It only remains to show that (10) can be produced from (9), (7), and (2), as we have already done in § 3. We then have, for the determination of y, z, p, q, r, s, t, as functions of x which shall render the integral I' a minimum (or I under the conditions (2), (3), (4), (5)), the necessary conditions (7), (9) [or (10)], and (2), (3), (4), (5), which latter also result from (6) on proper elimination; and these six exhaust the first necessary conditions for the solution of the given problem. But these equations (7), (9), (2), (3), (4), (5) are exactly the equations (14) of § 3 for the characteristic torsion strips. In other words, *the solutions in y, z, p, q, r, s, t, as functions of*

x, *of the equations* (14), *are precisely the only possible solutions of the given problem of the Calculus of Variations* ; *or, the only possible solutions of the given problem of the Calculus of Variations are given by the characteristic torsion strips of the equation* (2), *regarded as a differential equation.*

We might now easily go on to set up, as we have done in Part I, § 6 for equations of the first order, the equations of the common characteristics of two partial differential equations of the second order, and the condition that they be in involution, regarding the common characteristics as the common possible solutions of two problems in the Calculus of Variations, which reduce to a single problem of the type given above with one additional auxiliary condition of the type (2). Finally, we might seek the characteristics for two differential equations of the first order in two dependent variables, and so on. These problems will, however, offer no essential difficulty to the reader, and we will not enter into a discussion of them here ; the main point of the existence of a connection between the theory of characteristics and the Calculus of Variations already having been demonstrated.

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ERRATUM.

Page 139, line 7: instead of $q = \psi'(y)$, read $q = \phi'(y)$.

ON THE UNIFORMITY OF THE CONVERGENCE OF CERTAIN ABSOLUTELY CONVERGENT SERIES

BY MAXIME BÔCHER

IF the series

$$(1) \qquad u_1(x) + u_2(x) + \cdots$$

is absolutely and uniformly convergent for the values of x in a certain interval, and if we rearrange the terms, will the resulting series necessarily be uniformly convergent? This question must be answered in the negative as the following example shows :

$$(2) \qquad x^2 - x^2 + \frac{x^2}{1+x^2} - \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} + \cdots .$$

Here $\qquad S_{2n} = 0, \qquad S_{2n+1} = \dfrac{x^2}{(1+x^2)^n}\,.$